COMM 377: Part A – Synthesis

Harjoat S. Bhamra

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No – arbitrage pricing

- Instead of trying to determine asset prices in terms of fundamentals such as macroeconomic variables and investor preferences via an equilibrium approach, we shall take the prices of some assets as given and determine the prices of other assets in terms of these primitive assets assuming there are no arbitrage opportunities.
- Theoretically – equilibrium approach more appealing
- Practically – no-arbitrage approach more appealing
- Still need some theory to understand how the no arbitrage approach works
  - relation between equilibrium and no – arbitrage approach
    - certainty equivalents and risk aversion
  - risk- adjusted probabilities
    - an application
  - option pricing

Relation between equilibrium and no – arbitrage approach

Equilibrium approach

- Suppose the current date is t and you know you will receive 1 FC at date T in the future. What is the PV value of this cash flow in units of HC?
  - At date T, 1 FC is equivalent to 1 FC $\times \tilde{S}_T \frac{HC}{FC} = \tilde{S}_T$ HC

  What is the PV of $\tilde{S}_T$ HC?

$$\frac{E_t[\tilde{S}_T]}{1 + r_{t,T} + \underbrace{A_{t,T}}_{\text{risk adj.}}} HC \tag{1}$$

- We could try and determine the risk adjustment via equilibrium arguments.
- Does not work so well in practice.

No-arbitrage approach

- Is there another way to compute the PV in HC (at date t) of 1 FC to be received at date T?
  - Find PV of 1 FC in units of FC and then convert into HC at date t

$$\frac{1FC}{\underbrace{1+r^*_{t,T}}_{\text{1.Discount FC value}}} \times \underbrace{S_t \frac{HC}{FC}}_{\text{2.Convert FC present value into HC}} = \frac{S_t}{1+r^*_{t,T}} HC \tag{2}$$

  - Recall CIP: $F_{t,T} = S_t \frac{1+r_{t,T}}{1+r^*_{t,T}}$
  - CIP allows us to rewrite the PV expression in (2) as $\frac{F_{t,T}}{1+r_{t,T}} HC$
  - Compare the above expression with the original expression $\frac{E_t[\tilde{S}_T]}{1+r_{t,T}+\underbrace{A_{t,T}}_{\text{risk adj.}}} HC$
  - The two expressions must be equal

$$\frac{E_t[\tilde{S}_T]}{1+r_{t,T}+A_{t,T}} HC = \frac{F_{t,T}}{1+r_{t,T}} HC \tag{3}$$

Comparing the equilibrium and no-arbitrage approaches

$$\frac{E_t[\tilde{S}_T]}{1 + r_{t,T} + \underbrace{A_{t,T}}_{\text{risk adj.}}} HC = \frac{F_{t,T}}{\underbrace{1 + r_{t,T}}_{\text{no risk adj. in denom.}}} HC \tag{4}$$

- On the LHS, we know the risk adjustment adjusts the PV downwards because of risk in the future spot rate.
- Where is the risk adjustment on the RHS?
- Recall that the forward rate $F_{t,T}$ is the risk-free, i.e. certain HC cash flow which is equivalent to 1 FC.

$$F_{t,T} = CEQ_t[\tilde{S}_T] \tag{5}$$

- We now know where the risk adjustment on the RHS is!

$$\frac{E_t[\tilde{S}_T]}{1 + r_{t,T} + A_{t,T}} HC = \frac{\overbrace{CEQ_t[\tilde{S}_T]}^{\text{risk adj.}}}{1 + r_{t,T}} HC \tag{6}$$

Equilibrium v. no-arbitrage approach

- Why is it easier to use $\frac{CEQ_t[\tilde{S}_T]}{1+r_{t,T}}HC$ in practical applications instead of $\frac{E_t[\tilde{S}_T]}{1+r_{t,T}+A_{t,T}}HC$?
- Because $CEQ_t[\tilde{S}_T] = F_{t,T}$ is an observable market price, whereas $E_t[\tilde{S}_T]$ and $A_{t,T}$ are not.
- No need to use utility functions to compute $CEQ_t[\tilde{S}_T]$!

Why bother with utility functions?

- Why did we bother introducing utility functions at all if $CEQ_t[\tilde{S}_T] = F_{t,T}$?
- By using utility functions, we learnt that the certainty equivalent of a risky payoff is less than its expectation, if an investor is risk averse. Also, the more risk averse the investor the smaller the certainty equivalent is.

$$CEQ_t[\tilde{S}_T] = E_t[\tilde{S}_T] - \text{some downward risk adj.} \tag{7}$$

- Certainty equivalents are also referred to risk-adjusted or risk-neutral expectations.
- Gives rise to the idea of a risk-adjusted or risk-neutral probability – very powerful idea in practical applications, particularly asset valuation.

Risk-adjusted probabilities

- Why would we want to compute risk-adjusted probabilities?
- We know that the PV at date t (in HC) of $\tilde{S}_T$ received at date T is

$$PV_t[\tilde{S}_T] = \frac{CEQ_t[\tilde{S}_T]}{1 + r_{t,T}} \tag{8}$$

  - Compute the risk-adjusted expectation of a risky HC, date T payoff
  - Discount it at the HC risk-free rate.

- European call option payoff $\max(\tilde{S}_T - K, 0)$

$$PV_t[\max(\tilde{S}_T - K, 0)] = \frac{CEQ_t[\max(\tilde{S}_T - K, 0)]}{1 + r_{t,T}} \tag{9}$$

- If we know the risk - adjusted probability distribution for $\tilde{S}_T$, can compute the risk-adjusted expectation, $CEQ_t[\max(\tilde{S}_T - K, 0)]$.

Inferring risk-adjusted probabilities from market data

- We use forward rates to infer risk-adjusted probabilities for the future spot rate

$$F_{t,T} = CEQ_t[\tilde{S}_T] \tag{10}$$

- Assume $\tilde{S}_T$ can take two possible values: $S_u > S_d$.
- Denote risk-adjusted probability of future spot being S_d by q.

$$F_{t,T} = \underbrace{\overbrace{(1-q)S_u + qS_d}^{=CEQ_t[\tilde{S}_T],\text{ risk adj. expectation}}}_{\text{compute risk adj. expecation via risk adj. prob.'s}} \tag{11}$$

  - Solve for q in terms of market data ($F_{t,T}$) and assumptions about where future spot rate could be.

Modelling application

- Write future spot rate as the forward rate multiplied by some random variable, with a risk -adj. expectation of one, i.e.

$$\tilde{S}_T = F_{t,T}\tilde{M}_{t,T}, \quad CEQ_t[\tilde{M}_{t,T}] = 1. \tag{12}$$

- Why is this so?

$$\begin{aligned}
CEQ_t[\tilde{S}_T] &= CEQ_t[F_{t,T}\tilde{M}_{t,T}] \\
CEQ_t[\tilde{S}_T] &= F_{t,T}CEQ_t[\tilde{M}_{t,T}] \\
F_{t,T} &= F_{t,T}CEQ_t[\tilde{M}_{t,T}] \\
1 &= CEQ_t[\tilde{M}_{t,T}] \\
CEQ_t[\tilde{M_{t,T}}] &= 1
\end{aligned}$$

Two common choices for $\tilde{M}_{t,T}$

- $\tilde{M}_{t,T}$ can take two possible values: $M_u > M_d$.
  - Easy to find risk-adjusted probability that $\tilde{M}_{t,T}$ equals M_d

$$CEQ_t[\tilde{M}_{t,T}] = (1-q)M_u + qM_d = 1. \tag{13}$$

- $\widetilde{M}_{t,T} = e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(\widetilde{Z}_T - Z_t)}$, where the risk-adjusted probability distribution of $\widetilde{Z}_T - Z_t$ is given by $\widetilde{Z}_T - Z_t \sim N(0, \underbrace{T-t}_{variance})$.

$$\begin{aligned} CEQ_t[\widetilde{M}_{t,T}] &= CEQ_t[e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(\widetilde{Z}_T - Z_t)}] \\ &= CEQ_t[e^{-\frac{1}{2}\sigma^2(T-t)} e^{\sigma(\widetilde{Z}_T - Z_t)}] \\ &= e^{-\frac{1}{2}\sigma^2(T-t)} CEQ_t[e^{\sigma(\widetilde{Z}_T - Z_t)}] \\ &= e^{-\frac{1}{2}\sigma^2(T-t)} e^{\frac{1}{2}\sigma^2(T-t)} \\ &= 1 \end{aligned}$$

Option pricing

- Assume spot rates have following risk-adj. prob. distribution

$$\begin{aligned} \tilde{S}_T &= F_{t,T}\, \overbrace{e^{-\frac{1}{2}\sigma(T-t)+\sigma(\widetilde{Z}_T - Z_t)}}^{=\widetilde{M}_{t,T}}, \\ \widetilde{Z}_T - Z_t &\sim N(0, \underbrace{T-t}_{variance}) \end{aligned}$$

- σ is the annualized volatility of percentage changes in the spot rate
- Can evaluate $CEQ_t[\max(\widetilde{S}_T - K, 0)]$ explicitly – obtain a Black - Scholes type formula for a European style FX option